Exercises

Eigenvectors and Eigenvalues – Solutions

Exercise 1.

$$A \cdot a_1 = \begin{pmatrix} 0 \\ -6 \\ 8 \end{pmatrix} = 2 \cdot a_1 \Rightarrow \underline{\underline{\lambda_1 = 2}}$$

$$A \cdot a_2 = \begin{pmatrix} 35 \\ 28 \\ 21 \end{pmatrix} = 7 \cdot a_2 \Rightarrow \underline{\underline{\lambda_2 = 7}}$$

$$A \cdot a_3 = \begin{pmatrix} 24 \\ 6 \\ 8 \end{pmatrix} \neq \lambda \cdot a_3 \qquad a_3 \text{ is not an eigenvector}$$

To obtain the third eigenvector, we recall that the trace $\mathrm{tr}(A)$ is the sum of all eigenvalues, i.e.e $\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$. Thus

$$\lambda_3 = \mathrm{tr}(A) - \lambda_1 - \lambda_2 = (2 + 2 + 2) - 2 - 7 = -3.$$

Exercise 2. For A, we have $\det(A) = 5$, i.e. A^{-1} exists.

<u>Computation of the eigenvalues of A and A^{-1}</u>

- For $A = \begin{pmatrix} 4 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}$ we get the characteristic polynomial

$$\begin{aligned} P_A(\lambda) &= |A - \lambda E| = \begin{vmatrix} 4-\lambda & \sqrt{3} \\ \sqrt{3} & 2-\lambda \end{vmatrix} \\ &= (4-\lambda)(2-\lambda) - 3 = 8 - 6\lambda + \lambda^2 - 3 \\ &= \lambda^2 - 6\lambda + 5 = (\lambda - 5)(\lambda - 1) \end{aligned}$$

  The roots $\lambda_1 = 5$, $\lambda_2 = 1$ of $P_A(\lambda)$ are the eigenvalues of A.

- If x is a eigenvector for the eigenvalue λ of A then we have

$$Ax = \lambda x \Leftrightarrow \underbrace{A^{-1}A}_{I_n} x = A^{-1}\lambda x = \lambda A^{-1} x$$

$$\Leftrightarrow \quad x = \lambda A^{-1}x \quad \Leftrightarrow \quad A^{-1}x = \frac{1}{\lambda}x$$

Thus, if x is an eigenvector for the eigenvalue λ for A, then x is also an eigenvector for A^{-1} for the eigenvalue $\frac{1}{\lambda}$. Hence, the eigenvalues of A^{-1} are $\mu_1 = \frac{1}{5}$ and $\mu_2 = 1$.

Computation of the eigenvectors of A

We want to find the solutions of $(A - \lambda E)x = 0$ with $x \neq 0$.

- $\lambda_1 = 5 \Longrightarrow \begin{pmatrix} 4-5 & \sqrt{3} \\ \sqrt{3} & 2-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

x_1	x_2	r. S.
-1	$\sqrt{3}$	0
$\sqrt{3}$	-3	0
1	$-\sqrt{3}$	0
0	0	0

$$\Longrightarrow \quad x_1 = \sqrt{3}\,x_2$$

Solution:

$$x_2 = t \quad \Rightarrow \quad x = t\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, t \in \mathbb{R}, t \neq 0$$

- $\lambda_2 = 1 \Longrightarrow \begin{pmatrix} 4-1 & \sqrt{3} \\ \sqrt{3} & 2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

x_1	x_2	r. S.
3	$\sqrt{3}$	0
$\sqrt{3}$	1	0
1	$\frac{1}{\sqrt{3}}$	0
0	0	0

$$\Longrightarrow \quad x_1 = -\frac{1}{\sqrt{3}}x_2$$

$$x_2 = t \quad \Rightarrow \quad x = t\begin{pmatrix} -1/\sqrt{3} \\ 1 \end{pmatrix}, t \in \mathbb{R}, t \neq 0$$

Computation of the eigenvectors of A^{-1}

The eigenvectors of A^{-1} are the same as for A.

Exercise 3. We have to solve

$$0 \overset{!}{=} |A - \lambda E| = \det \begin{pmatrix} a-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & a-\lambda \end{pmatrix} = (a-\lambda)^2(1-\lambda)$$

We get the eigenvalues $\lambda_{1/2} = a$, $\lambda_3 = 1$ with the algebraic multiplicities 2 and 1.

$$0 \overset{!}{=} |B - \lambda E| = \det \begin{pmatrix} -\lambda & 0 & a \\ 0 & 1-\lambda & 0 \\ a & 0 & -\lambda \end{pmatrix} = (-\lambda)^2(1-\lambda) - a^2(1-\lambda)$$

$$= (1-\lambda)(\lambda^2 - a^2) = (1-\lambda)(\lambda+a)(\lambda-a)$$

We get the eigenvalues $\lambda_1 = 1$, $\lambda_2 = -a$, $\lambda_3 = a$.

Exercise 4.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Computation of eigenvalues

$$0 \overset{!}{=} \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = (2-\lambda+1)(2-\lambda-1)$$

$$= (3-\lambda)(1-\lambda)$$

$$\Longrightarrow \lambda_1 = 3 , \quad \lambda_2 = 1$$

Computation of eigenvectors

- $\lambda_1 = 3$

$$\Longrightarrow \begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow$$

x_1	x_2	r. S.
-1	1	0
1	-1	0

$$\Longrightarrow x_1 = x_2$$

Normalized vector: $\hat{x}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- $\lambda_2 = 1$

$$\implies \begin{pmatrix} 2-1 & 1 \\ 1 & 2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \implies \quad x_1 = -x_2$$

Normalized vector: $\hat{x}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Thus we get

$$A = X \Lambda X^{-1}$$

with $\quad \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad X^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

With $A^5 = X\Lambda^5 X^{-1}$ we can compute

$$\begin{aligned}
\implies A^5 &= \left(\frac{1}{\sqrt{2}}\right)^2 \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}^5 \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 243 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 243 & -1 \\ 243 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 244 & 242 \\ 242 & 244 \end{pmatrix} = \begin{pmatrix} 122 & 121 \\ 121 & 122 \end{pmatrix}
\end{aligned}$$

Exercise 5. The matrix A has the eigenvalue $\lambda = 0$ if and only if

$$0 = \det(A - \lambda I_n) = \det(A).$$

Since A is invertible if and only if $\det(A) \neq 0$ the claim follows.

Exercise 6.

1. The matrix P is

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}.$$

2. (a) The distribution after one day is

$$v_1 = P \cdot v_0 = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{16} \\ \frac{9}{16} \end{pmatrix}$$

(b) The distribution after two days is

$$v_2 = P \cdot v_1 = P \cdot P \cdot v_0 = \begin{pmatrix} \frac{1}{4} \\ \frac{7}{64} \\ \frac{41}{64} \end{pmatrix}$$

(c) We want to compute

$$v_{100} = P^{100} \cdot v_0.$$

With the diagonalization $P = VDV^{-1}$ we can compute the matrix power as

$$P^{100} = VD^{100}V^{-1}.$$

So we compute the eigenvalues of P:

$$\begin{aligned}
P_A(\lambda) &= \det \left[\begin{pmatrix} \frac{1}{4} - \lambda & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} - \lambda & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{4} - \lambda \end{pmatrix} \right] \\
&= \left(\frac{1}{4} - \lambda\right)\left(\frac{1}{4} - \lambda\right)\left(\frac{3}{4} - \lambda\right) + 0 + \frac{1}{32} - \frac{1}{8}\left(\frac{1}{4} - \lambda\right) - 0 - \frac{1}{16}\left(\frac{3}{4} - \lambda\right) \\
&= -\lambda\left(\lambda^2 - \frac{5}{4}\lambda + \frac{1}{4}\right) \\
&= -\lambda\left(\lambda - \frac{1}{4}\right)(\lambda - 1)
\end{aligned}$$

So the Eigenvalues are $0, \frac{1}{4}, 1$. The eigenvectors are

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{3}{8} \\ \frac{1}{8} \\ 1 \end{pmatrix}$$

and thus with

$$V = \begin{pmatrix} -1 & 0 & \frac{3}{8} \\ 1 & -1 & \frac{1}{8} \\ 0 & 1 & 1 \end{pmatrix} \text{ and } V^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

we have

$$P = V \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} V^{-1}$$

and can compute

$$P^{100} = V \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4^{100}} & 0 \\ 0 & 0 & 1 \end{pmatrix} V^{-1} \approx V \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} V^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

and finally get

$$P^{100} v_0 \approx \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} 0.5 \\ 0.25 \\ 0.25 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{12} \\ \frac{2}{3} \end{pmatrix}$$

3. We want a vector π with $P\pi = \pi$, i.e. an eigenvector for the eigenvalue $\lambda = 1$. From the previous part we know that

$$v = \begin{pmatrix} \frac{3}{8} \\ \frac{1}{8} \\ 1 \end{pmatrix}$$

is an eigenvalue for $\lambda = 1$. Since all proportions need to sum up to 1 we need to scale this vector by $\frac{2}{3}$ and obtain the vector

$$\pi = \frac{2}{3} v = \begin{pmatrix} \frac{2}{8} \\ \frac{1}{12} \\ \frac{2}{3} \end{pmatrix}.$$